

Rational solutions from Padé approximants for the generalized Hunter-Saxton equation

H Aratyn¹, J F Gomes², D V Ruy² and A H Zimerman²

¹ Department of Physics, University of Illinois at Chicago, 845 W. Taylor St., Chicago, IL 60607-7059

²Instituto de Física Teórica-UNESP, Rua Dr Bento Teobaldo Ferraz 271, Bloco II, 01140-070, São Paulo, Brazil

E-mail: jfg@ift.unesp.br

Abstract. Exact rational solutions of the generalized Hunter-Saxton equation are obtained using Padé approximant approach for the traveling-wave and self-similarity reduction. A larger class of algebraic solutions are also obtained by extending a range of parameters within the solutions obtained by this approach.

1. Introduction

The generalized Hunter-Saxton (HS) equation was originally proposed in [4] as

$$u_{xt} = uu_{xx} + ku_x^2. \quad (1)$$

This model reproduces the original HS equation when $k = 1/2$, which is a model for wave propagation in a director field of nematic liquid crystals. Besides, equation (1) appears in different applications, such as the study of the Einstein-Weyl space for $k = -1/3$ [3]. Solutions of (1) in a non-closed form were obtained in ref. [4] by a hodographic transformation, however the task of expressing these solutions in a closed form of the original variables is complicated and indirect.

It is known that the Padé approximant usually gives a good representation of the solution near a regular point (see [1] for more details) and it has widely been used to construct approximative solutions of differential equations with already determined initial conditions [6, 7]. Moreover, an exact solution was found for the Boussinesq equation in [7] for a predetermined initial condition. In this paper we employ the inverse path for study the generalized HS equation. We start with a rational expression given by the Padé approximant and seek for a set parameters or initial conditions that allows the ansatz to become an exact solution. Although there is no guarantee that a given ODE has rational or polynomial solutions, this approach provides a useful tool for searching for them.

The Padé approximant consists of a ratio of two polynomials that can be expressed as

$$[L/M](x) \equiv \frac{P_L(x)}{Q_M(x)} = \frac{\sum_{j=0}^L p_j x^j}{1 + \sum_{j=1}^M q_j x^j}$$

where the polynomials $P_L(x)$ and $Q_M(x)$ are defined such that $[L/M](x)$ agrees with the Taylor expansion a function $f = f(x)$ up to degree $L + M$, i.e.

$$f(x) = \frac{P_L(x)}{Q_M(x)} + \mathcal{O}(x^{L+M+1}).$$

Thus, in order to determine the Padé approximant of a Taylor expansion at $x = 0$ given by $f = \sum_{j=0}^{\infty} a_j x^j$, we need to solve a system of $L + M$ equations, namely,

$$\sum_{m+n=j} a_m q_n - p_j = 0, \quad j = 0, 1, \dots, L + M.$$

Consider now an arbitrary ordinary differential equation (ODE):

$$E(z, v, v_z, \dots; \mathcal{S}_0) = 0, \quad \mathcal{S}_0 \equiv \text{Cartesian product of the sets of parameters}, \quad (2)$$

and let $v = v(z; \mathcal{S}_0 \times \mathcal{S}_1)$ be the general solution of this equation, where \mathcal{S}_1 is the Cartesian product of the sets of initial conditions. Let us assume the origin has a regular local representation, i.e. $v = \sum_{j=0}^{\infty} v_j z^j$. Substituting in the differential equation (2), we have

$$E(v; \mathcal{S}_0) = \sum_{j=0}^{\infty} E_j(v_0, \dots, v_{j+n}) z^j = 0,$$

where n is the order of the equation. The above expression determines the coefficients v_j such that

$$v = \sum_{j=0}^{\infty} v_j(\mathcal{S}_0 \times \mathcal{S}_1) z^j.$$

Calculating the Padé approximant of this Taylor expansion, we have

$$v = \frac{P_L(z; \mathcal{S}_0 \times \mathcal{S}_1)}{Q_M(z; \mathcal{S}_0 \times \mathcal{S}_1)} + \mathcal{O}(z^{L+M+1}).$$

Let us assume now that there are subsets $\hat{\mathcal{S}}_0 \subset \mathcal{S}_0$ and $\hat{\mathcal{S}}_1 \subset \mathcal{S}_1$, such that the Padé approximant of $\hat{\mathcal{S}} = \hat{\mathcal{S}}_0 \times \hat{\mathcal{S}}_1$ leads to an exact solution, i.e.

$$v(z; \hat{\mathcal{S}}) = \frac{P_L(z; \hat{\mathcal{S}})}{Q_M(z; \hat{\mathcal{S}})}. \quad (3)$$

This ansatz allows us to seek solutions in the rational form without a need to consider further properties of the differential equation. Substituting (3) in (2), we have

$$E(z, v, v_z, \dots; \hat{\mathcal{S}}_0) = \frac{\sum_{j=0}^{\Lambda_{L,M}} \hat{E}_j(\hat{\mathcal{S}}) z^j}{D(\hat{\mathcal{S}})} = 0, \quad (4)$$

where the degree $\Lambda_{L,M}$ depends of L , M and the particular form of the differential equation. In order to determine all elements $s_j \in \hat{\mathcal{S}}$ for which the ansatz (3) is true, we need to solve a system of algebraic equations,

$$\hat{E}_j(\hat{\mathcal{S}}) = 0, \quad j = 0, 1, \dots, L + M - n \quad (5)$$

$$\hat{E}_j(\hat{\mathcal{S}}) = 0, \quad j = L + M - n + 1, \dots, \Lambda_{L,M} \quad (6)$$

$$D(\hat{\mathcal{S}}) \neq 0 \quad (7)$$

Observe that the system (5) is identically satisfied by the definition of the Padé approximant. Thus, the exact solution (3) is determined by the choice of parameters $\hat{\mathcal{S}}_0$ and initial conditions $\hat{\mathcal{S}}_1$.

In the next section, we apply this approach to the traveling-wave and self-similarity reductions of the generalized Hunter-Saxton equation. Moreover, for these specific reductions, we are able to extend the rational solutions obtained by the Padé approximant method to a more general set of parameters.

2. Generalized Hunter-Saxton equation

2.1. Traveling-wave reduction of the generalized HS equation

Applying the traveling-wave reduction

$$z = x + \mu t, \quad u(x, t) = u(z), \quad \mu = \text{constant} \quad (8)$$

to the generalized Hunter-Saxton equation (1) yields:

$$-\mu u_{zz} + uu_{zz} + ku_z^2 = 0. \quad (9)$$

The Taylor expansion of a solution to the above equation near the origin ($z = 0$) can be written as

$$u = u_0 + u_1 z + \frac{ku_1^2}{2(\mu - u_0)} z^2 + \frac{k(2k+1)u_1^3}{6(\mu - u_0)^2} z^3 + \frac{k(6k^2 + 7k + 2)u_1^4}{24(\mu - u_0)^3} z^4 + \dots \quad u_0 \neq \mu,$$

where u_0 and u_1 are arbitrary constants. In terms of the formalism from the previous section we have $\mathcal{S}_0 = (k, \mu)$ and $\mathcal{S}_1 = (u_0, u_1)$.

We now apply the Padé approximant approach to the two simplest cases $[1/1]$ and $[2/2]$. Let us begin with the ansatz

$$u \equiv u(z; (k, \mu, u_0, u_1)) = \frac{P_1(z; \hat{\mathcal{S}})}{Q_1(z; \hat{\mathcal{S}})} \quad (10)$$

where

$$\begin{aligned} P_1(z; \hat{\mathcal{S}}) &= u_0 + \frac{(2\mu - (k+2)u_0)u_1}{2(\mu - u_0)} z \\ Q_1(z; \hat{\mathcal{S}}) &= 1 - \frac{ku_1}{2(\mu - u_0)} z \end{aligned}$$

For this ansatz, the system (6) is given by

$$\hat{E}_1(\hat{\mathcal{S}}) = 8k(k+2)u_1^3(\mu - u_0)^3 = 0, \quad (11)$$

with $\Lambda_{1,1} = 1$. The condition (7) reads here as

$$D(\hat{\mathcal{S}}) = k^4 u_1^4 z^4 - 8k^3 u_1^3 (\mu - u_0) z^3 + 24k^2 u_1^2 (\mu - u_0)^2 z^2 - 32k u_1 (\mu - u_0)^3 z + 16(\mu - u_0)^4 \neq 0. \quad (12)$$

We see that $k = -2$ and $k = 0$ simultaneously solve conditions (11) and (12). Thus, by substituting these particular values of parameters on (10), we obtain exact solutions:

$$k = -2 \quad ; \quad u = \frac{(\mu - u_0)u_0 + z\mu u_1}{\mu - u_0 + u_1 z} \quad (13)$$

$$k = 0 \quad ; \quad u = u_0 + u_1 z. \quad (14)$$

Let us now take as an ansatz the Padé approximant of type $[2/2]$:

$$u \equiv u(z; (k, \mu, u_0, u_1)) = \frac{P_2(z; \hat{\mathcal{S}})}{Q_2(z; \hat{\mathcal{S}})}, \quad (15)$$

where

$$\begin{aligned} P_2(z; \hat{\mathcal{S}}) &= u_0 + \frac{(2\mu - (2k+3)u_0)u_1}{2(\mu - u_0)}z + \frac{(-6(k+1)\mu - (2k^2 + 7k + 6)u_0)u_1^2}{12(\mu - u_0)^2}z^2 \\ Q_2(z; \hat{\mathcal{S}}) &= 1 - \frac{(2k+1)u_1}{2(\mu - u_0)}z + \frac{k(2k+1)u_1^2}{12(\mu - u_0)^2}z^2. \end{aligned}$$

It follows that for this ansatz the system (6) is given by

$$\hat{E}_3(\hat{\mathcal{S}}) = 576k(k+2)(2k+1)(2k+3)u_1^5(\mu - u_0)^5 = 0 \quad (16)$$

$$\hat{E}_4(\hat{\mathcal{S}}) = -432k(k+1)(k+2)(2k+1)(2k+3)u_1^6(\mu - u_0)^4 = 0 \quad (17)$$

$$\hat{E}_5(\hat{\mathcal{S}}) = 24k(k+2)(2k+1)^2(2k+3)^2u_1^7(\mu - u_0)^3 = 0 \quad (18)$$

supplemented by condition (7), i. e.

$$\begin{aligned} D(\hat{\mathcal{S}}) &= k^4(2k+1)^4u_1^8z^8 - 24k^3(2k+1)^4u_1^7(\mu - u_0)z^7 \\ &+ 24k^2(2k+1)^3(20k+9)u_1^6(\mu - u_0)^2z^6 - 864k(2k+1)^3(3k+1)u_1^5(\mu - u_0)z^5 \\ &+ 432(2k+1)^2(38k^2 + 24k + 3)u_1^4(\mu - u_0)^4z^4 \\ &- 10368(2k+1)^2(3k+1)u_1^3(\mu - u_0)z^3 + 3456(2k+1)(20k+9)u_1^2(\mu - u_0)z^2 \\ &- 41472(2k+1)u_1(\mu - u_0)^7z + 20736(\mu - u_0)^8 \neq 0 \end{aligned}$$

Substituting the solution of system (16-18) into the ansatz (15) yields exact solutions:

$$\begin{aligned} k = -2 \quad ; \quad u &= \frac{(\mu - u_0)u_0 + z\mu u_1}{\mu - u_0 + u_1z} \\ k = -3/2 \quad ; \quad u &= \frac{4(u_0 - \mu)u_0 + 4(\mu - u_0)\mu u_1z + \mu u_1^2z^2}{(2(\mu - u_0) + u_1z)^2} \\ k = -1/2 \quad ; \quad u &= u_0 + u_1z - \frac{u_1^2z^2}{4(\mu - u_0)} \\ k = 0 \quad ; \quad u &= u_0 + u_1z \end{aligned}$$

Notice that with the above choice of parameters the order of the Padé approximants is reduced accordingly.

Observe that solutions (13) and (14) obtained from the first ansatz are included in the above set of solutions. This is not a general case as it will be evident in the next section. These ansatzes can be worked out by an algebraic manipulation software such as Mathematica or Maple so we omit details for the system (5) and (6) in the remaining part of the text. Following similar procedure as above for the ansatz:

$$u \equiv u(z; (k, \mu, u_0, u_1)) = \frac{P_3(z; \hat{\mathcal{S}})}{Q_3(z; \hat{\mathcal{S}})}, \quad (19)$$

we obtain the solutions

$$\begin{aligned}
k = -2 \quad ; \quad u &= \mu + \frac{-\mu^2 + 2\mu u_0 - u_0^2}{\mu + u_1 z - u_0} \\
k = -3/2 \quad ; \quad u &= \mu - \frac{4(\mu^3 - 3\mu^2 u_0 + 3\mu u_0^2 - u_0^3)}{(2\mu + u_1 z - 2u_0)^2} \\
k = -4/3 \quad ; \quad u &= \mu - \frac{27(\mu^4 - 4\mu^3 u_0 + 6\mu^2 u_0^2 - 4\mu u_0^3 + u_0^4)}{(3\mu + u_1 z - 3u_0)^3} \\
k = -\frac{2}{3} \quad ; \quad u &= \frac{u_1^3 z^3}{27(\mu - u_0)^2} - \frac{u_1^2 z^2}{3(\mu - u_0)} + u_0 + u_1 z \\
k = -\frac{1}{2} \quad ; \quad u &= -\frac{u_1^2 z^2}{4(\mu - u_0)} + u_0 + u_1 z \\
k = 0 \quad ; \quad u &= u_0 + u_1 z
\end{aligned}$$

We can see that all non-trivial solutions of the above system are expressed in terms of $\hat{\mathcal{S}}_1 = \mathcal{S}_1$ and $\hat{\mathcal{S}}_0$ being independent of the initial conditions, therefore, they are general solutions.

A careful study of these solutions suggests a general expression that compactly expresses the whole set $\hat{\mathcal{S}}$. Indeed direct verification confirms that all solutions are reproduced by

$$u = \mu + \frac{(u_0 - \mu)}{(1 + \frac{(-1)^r z u_1}{r(-\mu + u_0)})^r}, \quad r = -\frac{1}{(k+1)}. \quad (20)$$

Although this solution was obtained originally for integer values of r only it can be checked that it holds for any r different from 0. The closed general solution for $k = 1/2$ was obtained recently in [5] and agrees with (20).

2.2. Self-similarity reduction of the generalized HS equation

In this section, we explore the self-similarity reduction of the generalized Hunter-Saxton equation. Observe that inserting the self-similarity reduction expression

$$u(x, t) = t^{-(1+\xi)} v(z), \quad z = xt^\xi, \quad \xi = \text{constant} \quad (21)$$

into (1) yields

$$v_z - \xi z v_{zz} + v v_{zz} + k v_z^2 = 0 \quad (22)$$

The local representation of the solution v around the origin is

$$v = v_0 + v_1 z - \frac{v_1(1 + k v_1)}{2v_0} z^2 + \frac{v_1(1 + k v_1)(1 - \xi + (1 + 2k)v_1)}{6v_0^2} z^3 + \dots, \quad (23)$$

where v_0 and v_1 are arbitrary constants, $\mathcal{S}_0 = (k, \xi)$ and $\mathcal{S}_1 = (v_0, v_1)$. Notice that for $v_1 = -1/k$ the above series truncates leading to the solution

$$v = v_0 - \frac{z}{k}.$$

We can apply the ansatz for the Padé approximants provided that $v_1 \neq -\frac{1}{k}$. Therefore, considering the ansatz $v \equiv v(z; (k, \xi, v_0, v_1)) = \frac{P_1(z; \hat{\mathcal{S}})}{Q_1(z; \hat{\mathcal{S}})}$, we have the set $\hat{\mathcal{S}}$ and the corresponding solution given by

$$(k, \xi, v_0, v_1) = (-2, -1/2, v_0, v_1) \quad ; \quad v = \frac{v_0(2v_0 + z)}{2v_0 + (1 - 2v_1)z}$$

Applying the same procedure to the ansatz $v \equiv v(z; (k, \xi, v_0, v_1)) = \frac{P_2(z; \hat{\mathcal{S}})}{Q_2(z; \mathcal{S})}$, it give us

$$(k, \xi, v_0, v_1) = (-2, 1, v_0, v_1) \quad ; \quad v = \frac{2v_0^2 - v_1 z^2}{2v_0 - 2v_1 z}$$

$$(k, \xi, v_0, v_1) = (-3/2, -1/3, v_0, v_1) \quad ; \quad v = \frac{2v_0[18u_0^2 + 12v_0 z + (2 - 3v_1)z^2]}{[6v_0 + (2 - 3v_1)z]^2}$$

$$(k, \xi, v_0, v_1) = (-1/2, 1, v_0, v_1) \quad ; \quad v = v_0 + v_1 z + \frac{v_1(v_1 - 2)}{4v_0} z^2$$

In order to obtain more solutions which helps to look for a compact expression of all rational solutions, we also consider the ansatz $v \equiv v(z; (k, \xi, v_0, v_1)) = \frac{P_3(z; \hat{\mathcal{S}})}{Q_3(z; \mathcal{S})}$. This yields the following solutions

$$(k, \xi, v_0, v_1) = (-3/2, 1, v_0, v_1) \quad ; \quad v = \frac{2(6v_0^3 - 3v_0 v_1 z^2 + v_1^2 z^3)}{3(2v_0 - z v_1)^2}$$

$$(k, \xi, v_0, v_1) = (-4/3, -1/4, v_0, v_1) \quad ; \quad v = -\frac{3v_0[576u_0^3 + 432v_0^2 z - 36v_0(4v_1 - 3)z^2 + (3 - 4v_1)^2 z^3]}{[(4v_1 - 3)z - 12v_0]^3}$$

$$(k, \xi, v_0, v_1) = (-2/3, 1/2, v_0, v_1) \quad ; \quad v = v_0 + v_1 z + \frac{v_1(2v_1 - 3)}{6v_0} z^2 + \frac{v_1(2v_1 - 3)^2}{108v_0^2} z^3$$

$$(k, \xi, v_0, v_1) = (-2/3, 1, v_0, v_1) \quad ; \quad v = v_0 + v_1 z + \frac{v_1(2v_1 - 3)}{6v_0} z^2 + \frac{v_1^2(2v_1 - 3)}{54v_0^2} z^3.$$

We see that the parameters of the above general solutions can be organized as

$$k = -\frac{(r+1)}{r} \quad \text{with} \quad \xi = 1 \quad \text{or} \quad \xi = -\frac{1}{(r+1)}$$

In order to express the solutions in a compact form, let us define the constants u_0, z_0 as

$$v_0 = \frac{z_0 v_1}{r}, \quad v_1 = \frac{r}{(r+1)} + \frac{(-1)^r r u_0}{z_0^{r+1}}$$

when $\xi = 1$ and

$$v_0 = \left(-\frac{1}{(r+1)} + \frac{v_1}{r} \right) z_0, \quad v_1 = \frac{(-1)^r r u_0}{z_0^{r+1}}$$

when $\xi = -1/(r+1)$.

The solutions found by the Padé approximant approach for the self-similarity reduction can be summarized as

$$v = \frac{u_0}{(z - z_0)^r} + \frac{z_0 + rz}{r+1}, \quad \text{when} \quad \xi = 1, \quad (24)$$

$$v = \frac{u_0}{(z - z_0)^r} - \frac{z_0}{r+1}, \quad \text{when} \quad \xi = -\frac{1}{(r+1)} \quad (25)$$

where $r = -1/(k+1)$. This expression was derived for an integer r , however, one can verify that expressions (24) and (25) are still solutions for a non-integer r (different from 0 and -1).

Acknowledgments

The authors thank CNPq and Fapesp for support.

References

- [1] Baker Jr G A and Graves-Morris P 1996 *Padé Approximants* (New York: Cambridge University Press)
- [2] Hunter J K and Saxton R 1991 *SIAM J. Appl. Math.* **51** 1498
- [3] Tod K P 2000 *J. Math. Phys.* **41** 5572
- [4] Pavlov M V 2001 *Theor. and Math. Phys.* **128** 927
- [5] Al-Ali E M 2013 *Int. Journal of Math. Analysis* **7** 1647
- [6] Boyd J P 1997 *Computers in Physics* **11** 299
- [7] Mousa M M and Kaltay A 2009 *Int. J. of Eng. and Applied Sciences* **5** 237